EXERCISES [MAI 5.10-5.13]
BASIC INTEGRALS – AREAS

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A. Paper 1 questions (SHORT)

INDEFINITE INTEGRALS

1. [Maximum mark: 20]

Find the following integrals

$\int x^9 dx$
$\int 20x^9 dx$
$\int x^{-9} dx$
$\int 16x^{-9} dx$
$\int (x^3 + x^2 + x + 3)dx$
$\int (4x^3 - 12x^2 + 6x + 3)dx$
$\int (x^{-3} + x^{-2} + 3)dx$
$\int (4x^{-3} - 12x^{-2} + 3)dx$
$\int e\,dx$
$\int \pi^2 dx$

2. [Maximum mark: 30]

Find the following integrals (modify the expressions first, if necessary).

$$\int\left(\frac{4}{x^3}-\frac{12}{x^2}+3\right)dx =$$

$$\int\left(\frac{8}{x^5}-\frac{12}{x^3}+6x\right)dx =$$

$$\int\left(\frac{x^3}{4}-\frac{2x^2}{3}-\frac{x}{7}+3\right)dx =$$

$$\int\left(\frac{1}{4x^3}-\frac{2}{3x^2}+3\right)dx =$$

$$\int\left(\frac{3}{4x^3}-\frac{2x^2}{3}\right)dx =$$

$$\int(x+1)(x+2)dx =$$

$$\int(2x^2+5x+1)(3x-4)dx =$$

$$\int\frac{2x^5+5x^3+x^2}{x^2}dx =$$

$$\int\frac{2x^3+5x+1}{x^3}dx =$$

$$\int\frac{2x^7+5x+4}{3x^3}dx =$$

3. [Maximum mark: 5]

Let $f'(x) = 6x^2 + 1$. Find $f(x)$, given that $f(0) = 3$.

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4. [Maximum mark: 7]

Let $f''(x) = 12x^2$.

(a) Find $f'(x)$, given that $f'(0) = 3$. [4]

(b) Find $f(x)$, given that $f(0) = 2$. [3]

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5. [Maximum mark: 4]

Let $f'(x) = 1 - x^2$. Given that $f(3) = 0$, find $f(x)$.

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6. [Maximum mark: 5]

It is given that $\dfrac{dy}{dx} = x^3 + 2x - 1$ and that $y = 13$ when $x = 2$. Find y in terms of x.

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7. [Maximum mark: 5]

The curve $y = f(x)$ passes through the point (2, 6). Given that $\dfrac{dy}{dx} = 3x^2 - 5$, find y in terms of x.

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8. [Maximum mark: 5]

A curve with equation $y = f(x)$ passes through the point (1, 1). Its gradient function is $f'(x) = -2x + 3$. Find the equation of the curve.

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DEFINITE INTEGRALS

9. [Maximum mark: 30]

Calculate the following definite integrals. Show your work **analytically**.

$\int_0^1 x^9 \, dx$
$\int_0^1 20x^9 \, dx$
$\int_1^2 x^{-2} \, dx$
$\int_1^2 8x^{-3} \, dx$
$\int_0^2 6x^2 \, dx$
$\int_1^2 6x^2 \, dx$
$\int_0^1 (2x+3) \, dx$
$\int_1^2 (2x+3) \, dx$
$\int_0^2 (2x+3) \, dx$
$\int_{-2}^2 (2x+3) \, dx$
$\int_0^{10} x \, dx$
$\int_0^{10} 5 \, dx$
$\int_0^{10} dx$
$\int_4^{10} dx$
$\int_a^b dx$

10. [Maximum mark: 6]

    (a)   Find $\int_{1}^{a}\left(3x^2+1\right)dx$ in terms of a.   [3]

    (b)   Find $\int_{1}^{b}\left(6x^2+\dfrac{1}{x^2}\right)dx$ in terms of b.   [3]

11. [Maximum mark: 6]

Find the real number $k>1$ for which $\int_{1}^{k}\left(1+\dfrac{1}{x^2}\right)dx=\dfrac{3}{2}$.

PROPERTIES OF DEFINITE INTEGRALS

(Questions 12-17 mainly for HL)

12. [Maximum mark: 20]

Let $\int_5^7 f(x)dx = 8$ and $\int_5^7 g(x)dx = 2$

Find the value of the following expressions

$\int_7^5 f(x)dx$
$\int_5^7 3f(x)dx$
$\int_5^7 \dfrac{f(x)}{4}dx$
$\int_5^7 (f(x)+1)dx$
$\int_5^7 (f(x)+x)dx$
$\int_5^6 f(x)dx + \int_6^7 f(x)dx$
$\int_5^7 [f(x)+g(x)]dx$
$\int_5^7 [f(x)-g(x)]dx$
$\int_5^7 [2f(x)+3g(x)]dx$
$\int_5^7 [f(x)-4g(x)]dx$

13. [Maximum mark: 6]

Given that $\int_1^3 g(x)dx = 10$, deduce the value of (i) $\int_1^3 \dfrac{1}{2}g(x)dx$; (ii) $\int_1^3 (g(x)+4)dx$.

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14. [Maximum mark: 6]

Let f be a function such that $\int_0^3 f(x)\,dx = 8$.

(a) Deduce the value of (i) $\int_0^3 2f(x)\,dx$ (ii) $\int_0^3 (f(x)+2)\,dx$ [4]

(b) Write down the value of the expression $\int_0^1 f(x)\,dx + \int_1^3 f(x)\,dx$. [2]

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15. [Maximum mark: 6]

Given that $\int_1^3 f(x)\,dx = 5$, deduce the value of (i) $\int_1^3 2f(x)\,dx$ (ii) $\int_1^3 (3x^2 + f(x))\,dx$.

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16. [Maximum mark: 6]

The table shows some values of two functions, f, g and of their derivatives f', g':

x	1	2	3	4
$f(x)$	5	4	−1	3
$g(x)$	1	−2	2	−5
$f'(x)$	5	6	0	7
$g'(x)$	−6	−4	−3	4

(a) Calculate $\dfrac{d}{dx}\left(f(x)+g(x)\right)$, when $x=4$; [2]

(b) Calculate $\displaystyle\int_{1}^{3}\left(g'(x)+6\right)dx$. [4]

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17. [Maximum mark: 7]

Let $\displaystyle\int_{1}^{5} 3f(x)\,dx=12.$

(a) Show that $\displaystyle\int_{5}^{1} f(x)\,dx=-4$ [3]

(b) Find the value of $\displaystyle\int_{1}^{2}\left(x+f(x)\right)dx+\int_{2}^{5}\left(x+f(x)\right)dx$ [4]

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AREAS

18. [Maximum mark: 4]

The diagram shows part of the graph of $y = 12x^2(1-x)$.
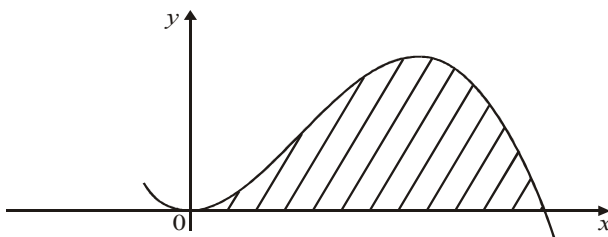
(a) Write down an integral which represents the area of the shaded region. [2]

(b) Find the area of the shaded region. [2]

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19. [Maximum mark: 5]

Let R be the region between the curve $y = -3x^2 + 6x$ and x-axis.

(a) Write down an integral which represents the area of the region R. [3]

(b) Find the area of the region R. [2]

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20. [Maximum mark: 7]

Let $f(x) = -3x^2 + 8x$ and $g(x) = 2x$. Let R be the region between the two curves.

(a) Sketch a diagram showing the graphs of f and g and shade the region R. [2]

(b) Write down an expression which represents the area of the region R. [3]

(c) Find the area of the region R. [2]

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21. [Maximum mark: 6]

Let $f(x) = -3x^2 + 8x$ and $g(x) = 2x$. Let S be the region enclosed by the two curves and x - axis.

(a) Write down an expression which represents the area of the region S. [4]

(b) Find the area of the region S. [2]

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22. [Maximum mark: 9]

(a) Find the area between the curve $y = 2 + x - x^2$ and x-axis. [3]

(b) Find the area between the curve $y = 3 + x - x^2$ and $y = 3 - 3x + x^2$ [4]

(c) Find the area between the curve $y = 2 + x - x^2$ and $y = 2 - 3x + x^2$ [2]

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23. [Maximum mark: 6]

The figure below shows part of the curve $y = x^3 - 7x^2 + 14x - 7$.

The curve crosses the x-axis at the points A, B and C.
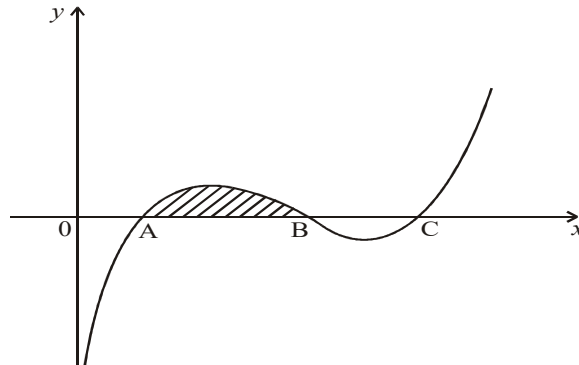
(a) Find the x-coordinate of A and the x-coordinate of B. [2]

(b) Write down an expression for the area of the shaded region. [2]

(c) Find the area of the shaded region. [2]

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24. [Maximum mark: 4]

Find the **total** area enclosed by the curve $y = x^3 - 7x^2 + 14x - 7$ and x-axis.

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TRAPEZOIDAL RULE

25. [Maximum mark: 10]

The region R between the curve $y = -3x^2 + 6x$ and x - axis is given by the definite integral

$$A = \int_{0}^{2} (6x - 3x^2)dx$$

(a)   Write down the exact value of A. [1]

(b)   Use the trapezoidal rule to find approximations for A.

(i)   with $n = 2$ intervals

(ii)   with $n = 4$ intervals

(iii) with interval length $h = 0.4$ [9]

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26. [Maximum mark: 9]

Let $f(x) = \ln x$

(a) Complete the following table of values, using 3 s.f.

x	1	1.5	2	2.5	3
$f(x)$			0.693	0.916	

[2]

(b) Find an approximation for the definite integral $\int_1^3 \ln x\, dx$ using the trapezoidal rule

    (i) with $n = 2$ intervals

    (ii) with $n = 4$ intervals

[5]

The exact value of the definite integral $\int_1^3 \ln x\, dx$ is $3\ln 3 - 2$.

(c) Find the percentage error for your estimation in question (b)(ii)

[2]

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B. Paper 2 questions (LONG)

27. [Maximum mark: 12]

The function f is such that $f''(x) = 2x - 2$.

When the graph of f is drawn, it has a minimum point at (3, –7).

(a) Show that $f'(x) = x^2 - 2x - 3$ and hence find $f(x)$. [6]

(b) Find $f(0)$, $f(-1)$ and $f'(-1)$ [3]

(c) Hence sketch the graph of f labelling it with the information obtained in part (b). [3]

(**Note:** It is **not** necessary to find the coordinates of the points where the graph cuts the x-axis.)

28. [Maximum mark: 14]

Consider the function f with second derivative $f''(x) = 3x - 1$. The graph of f has a minimum point at A(2, 4) and a maximum point at B.

(a) Given that $f'(x) = \dfrac{3}{2}x^2 - x + p$, show that $p = -4$. [4]

(b) Find $f(x)$. [7]

(c) Use $f'(x)$ to find the x-coordinate of B. [3]

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29. [Maximum mark: 15]

A curve has equation $y = x(x-4)^2$.

(a) Write down the x-intercepts of the curve. [2]

(b) Find $\dfrac{dy}{dx}$. [3]

(c) Use $\dfrac{dy}{dx}$ to find the coordinates of (i) the maximum point; (ii) the minimum point; [4]

(d) Use your answers above to sketch a graph of the curve for $0 \le x \le 4$ [3]

(e) (i) On your sketch indicate by shading the region whose area is given by the

following integral: $\displaystyle\int_0^4 x(x-4)^2 \, dx$

(ii) Explain, using your answer to part (a), why the value of this integral is

greater than 0 but less than 40. [3]

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30. [Maximum mark: 15]

The diagram below shows part of the graph of the function $f:x \mapsto -x^3 + 2x^2 + 15x$.

The graph intercepts the x-axis at A(−3,0), B(5,0) and the origin, O. There is a minimum point at P and a maximum point at Q.

(a) Write down the x- Intercepts of the graph. [2]

(b) (i) Find the **exact** values of x at which $f'(x) = 0$;

(ii) Find the value of the function at Q. [7]

(c) (i) Find the equation of the tangent to the graph of f at O.

(ii) This tangent cuts the graph of f at point R. Give the x-coordinate of R. [4]

(d) Determine the area of the shaded region. [2]

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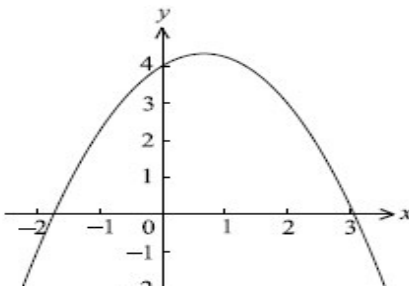
31. [Maximum mark: 12]

Let $f(x) = -\dfrac{3}{4}x^2 + x + 4$.

(a) (i) Find the equation of the normal to the curve of f at (2, 3).

(ii) This normal intersects the curve of f at (2, 3) and at one other point P.

Find the x-coordinate of P. [9]

Part of the graph of f is given below.

(b) Let R be the region under the curve of f from $x = -1$ to $x = 2$.

Write down an expression for the area of R and **hence** evaluate this area. [3]

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32. [Maximum mark: 16]

The function f is defined by $f:x \mapsto -0.5x^2 + 2x + 2.5$.

(a) Write down   (i) $f'(x)$;   (ii) $f'(0)$ [2]

(b) Let N be the normal to the curve at the point where the graph intercepts the y-axis. Show that the equation of N may be written as $y = -0.5x + 2.5$.   [3]

Let $g:x \mapsto -0.5x + 2.5$

(c) (i) Find the solutions of $f(x) = g(x)$.

     (ii) Hence find the coordinates of the other point of intersection of the normal and the curve.   [6]

(d) Let R be the region enclosed between the curve and N.

     (i) Write down an expression for the area of R.
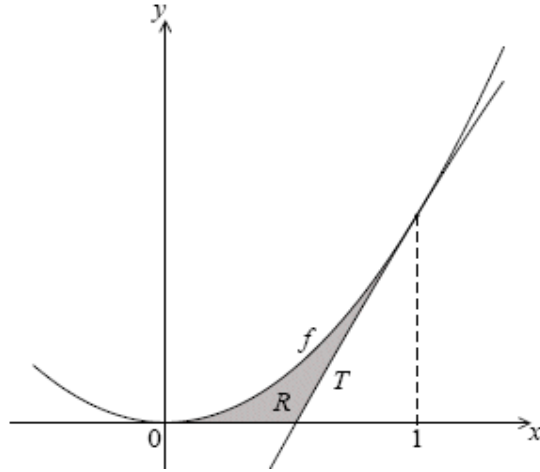
     (ii) Hence write down the area of R.   [5]

33. [Maximum mark: 16]

The following diagram shows part of the graph of the function $f(x) = 2x^2$.

The line *T* is the tangent to the graph of f at $x = 1$.

(a) Show that the equation of *T* is $y = 4x - 2$. [5]

(b) Find the x-intercept of *T*. [2]

(c) The shaded region *R* is enclosed by the graph of f, the line *T*, and the x-axis.

    (i) Write down an expression for the area of *R*.

    (ii) Find the area of *R*. [9]

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34. [Maximum mark: 10]

Let $f(x) = x(x-5)^2$, for $0 \le x \le 6$. The diagram 1 shows the graph of f.

Diagram 1

Diagram 2

Let R be the region enclosed by the x-axis and the curve of f.

(a) Find the area of R. [3]

(b) The diagram 2 shows a part of the graph of a quadratic function $g(x) = x(a-x)$.

The graph of g crosses the x-axis when $x = a$.

The area of the shaded region is equal to the area of R. Find the value of a. [7]

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